Chapter 10. Singularly Perturbed Impulsive-Switched Systems w/ Time Delay.

10.1 Problem Formulation.

(10.2)
$$\begin{cases} \dot{x} = f_{\sigma(t)}(t, x, x_t, z, z_t), & t \neq t_K \\ \varepsilon \dot{z} = g_{\sigma(t)}(t, x, x_t, z, z_t), & t \neq t_K \\ \Delta x = B_K x(t^-), & t = t_K \\ \Delta z = C_K z(t^-), & t = t_K \end{cases}$$

Impulse: $x(t_K) - x(t_K^-) = B_K x(t_K^-)$

Since $x(t_K^-) := \lim_{t \to t_K^-} x(t)$, $t < t_K$

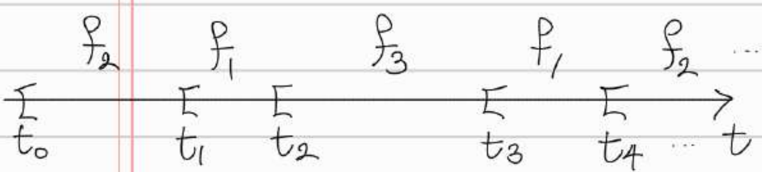
where $\sigma : [t_0, \infty) \to J := \{1, \cdots, N\} = J_u \cup J_s$

piecewise const. ftn. (N: # of subsystems)

$\sigma(t) := i_K$ if $t \in [t_{K-1}, t_K)$: discontinuous at each t_K.

$\Rightarrow \{t_K\}_{K=1}^{\infty}$: seq. of impulsive-switching times.

s.t. $t_{K-1} < t_K$, $\lim_{K \to \infty} t_K = \infty$



$i_1 = 2$, $i_2 = 1$, $i_3 = 3$, $i_4 = 1$, ...

On 4th subinterval $[t_3, t_4)$, 1st subsystem is activated.

* $T^+(t_0, t)$ ($T^-(t_0, t)$) : total activation time of unstable (stable) subsystem over the time interval $[t_0, t)$

<u>Note</u> $T^+(t_0, t) + T^-(t_0, t) =$ the length of $[t_0, t) = t - t_0$.

* <u>Assumption</u>

① $x(t_K^+) = x(t_K)$, $z(t_K^+) = z(t_K)$: the solution of the system is right continuous.

② f_{i_K}, g_{i_K} are smooth.

Thm 10.1 trivial solution of (10.2) is exponentially stable if

(A1) $i_K \in J_u \Rightarrow A_{11 i_K}$ has eigenvalues w/ positive real parts

$i_K \in J_s \Rightarrow A_{11 i_K}$ has " w/ negative " .

(A2) $\forall K \in N,\ t \in [t_{K-1}, t_K),\ \exists a_{▨}, b_{▨} > 0$ s.t. (chap.9)

① $2x^T P_{1 i_K} [A_{12 i_K} + \cdots] \le a_{11 i_K} \|x\|^2 + a_{12 i_K} \|x_t\|_\tau^2 + \cdots$, $= I_m$, $= I_n$

② $P_{1 i_K}, P_{2 i_K}$: Positive-definite matrix for given $Q_{1 i_K}, Q_{2 i_K}$

$\lambda_{1m} = \min\{\lambda_{min}(P_{1 i_K}) \mid K \in N\}$ (λ_{2m}, $P_{2 i_K}$)

$\lambda_{1M} = \max\{\lambda_{max}(P_{1 i_K}) \mid K \in N\}$ (λ_{2M}, $P_{2 i_K}$)

$\Rightarrow \mu = \max\left\{\frac{\lambda_{1M}}{\lambda_{1m}}, \frac{\lambda_{2M}}{\lambda_{2m}}\right\}$

(A3) : conditions are designed to apply Lem 2.3 and Lem 2.4,

$\exists r^* > 0$ s.t. $A_{11 i_K} - r^* I$: Hurwitz ($\forall K$)

(A4) : $\lambda^+ = \max\{\xi_{i_K} \mid i_K \in J_u\}$, $\lambda^- = \min\{\zeta_{i_K} \mid i_K \in J_s\}$ (ksi, zeta)

growth rate decay rate of subsystem

defined in Lem 2.3 and 2.4 (P.14)

(10.3) : $\exists \lambda^* \in (0, \lambda^-)$ s.t. $\inf_{t \ge t_0} \frac{T^-(t_0,t)}{T^+(t_0,t)} \ge \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*}$

(10.4) (i) for $K \in N$ satisfying $i_K \in J_u$,

$\ln \mu(\alpha_K + \beta_K + r_K + \psi_K) - \nu(t_K - t_{K-1}) \le 0$

$\Rightarrow \mu(\alpha_K + \beta_K + r_K + \psi_K) \le e^{\nu(t_K - t_{K-1})}$

(ii) for $K \in N$ satisfying $i_K \in J_s$,

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Proof

$$\begin{cases} V_i(t) := V_i(x(t)) = x^T(t)\, P_{1i}\, x(t) \\ W_i(t) := W_i((z-h_i)(t)) = (z-h_i)^T(t)\, P_{2i}\, (z-h_i(t)) \end{cases}$$

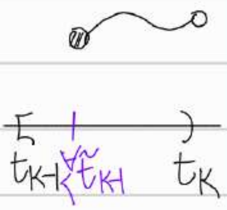
defined in A2.

where $h_i(t) := -B_{21i}^{\dagger}\,[A_{21}\, x + A_{22i}\, x_t]$

so that $B_{21i} h_i = A_{21i}\, x + A_{22i}\, x_t$.

(*) (i) If $i_k \in J_u$, $t \in (t_{k-1}, t_k)$ $\forall \tilde{t}_{k-1} \in (t_{k-1}, t_k)$,

this ineq. holds for $\forall t \in [\tilde{t}_{k-1}, t_k)$



$$\begin{pmatrix} \dot{V}(t) \\ \dot{W}(t) \end{pmatrix} \preceq \underbrace{\begin{pmatrix} \square & \triangle \\ \bigcirc & \star \end{pmatrix}}_{=\tilde{A}_i} \begin{pmatrix} V(t) \\ W(t) \end{pmatrix} + \underbrace{\begin{pmatrix} \square' & \triangle' \\ \bigcirc' & \star' \end{pmatrix}}_{=\tilde{B}_i} \begin{pmatrix} \|V_{it}\|_\tau \\ \|W_{it}\|_\tau \end{pmatrix}$$

$\|V_{it}\|_\tau = \sup_{t-\tau \le s \le t} \|V_{i_k}(s)\|$

$\xrightarrow{\text{A3(i), Lemma 2.4}}$ $\exists\, \xi_i > 0$ s.t. (ksi)

$$\begin{cases} V_{i_k}(t) \le (\|V_{i_k t_{k-1}}\|_\tau + \|W_{i_k t_{k-1}}\|_\tau)\, e^{\xi_i (t - t_{k-1})} \\ W_{i_k}(t) \le (\|V_{i_k, t_{k-1}}\|_\tau + \|W_{i_k, t_{k-1}}\|_\tau)\, e^{\xi_i (t - t_{k-1})} \end{cases}$$

$\|V_{i_k t_{k-1}}\|_\tau = \sup_{t_{k-1}-\tau \le s \le t_{k-1}} \|V_{i_k}(s)\|$

the right continuity of $x(t), z(t)$ ensures the right continuity of V and W.

for $\forall t \in [t_{k-1}, t_k)$

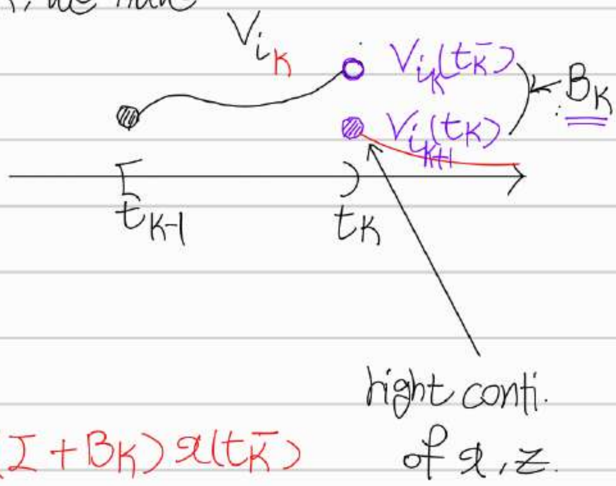
$\dot{x}(t) =$ [hatched box] $\quad t \ne t_k$.

(**) (ii) If $i_k \in J_s$, A(3)(ii), Lemma 2.3 $\exists\, \zeta_i > 0$ s.t. (zeta)

$$\begin{cases} V_{i_k}(t) \le (\|(V_{i_k})_{t_{k-1}}\|_\tau + \|(W_{i_k})_{t_{k-1}}\|_\tau)^{-\zeta_{i_k}(t - t_{k-1})} \\ W_{i_k}(t) \le (\|(V_{i_k})_{t_{k-1}}\|_\tau + \|(W_{i_k})_{t_{k-1}}\|_\tau)^{-\zeta_{i_k}(t - t_{k-1})} \end{cases}$$

for $\forall t \in [t_{k-1}, t_k)$

At the impulsive-switching moment $t = t_K$, we have



$$V_{i_{K+1}}(t_K) = x(t_K)^T P_{i_{K+1}} x(t_K)$$

$$= [(I+B_K)x(t_K^-)]^T P_{i_{K+1}} [(I+B_K)x(t_K^-)]$$

$\Delta x = x(t_K) - x(t_K^-) = B_K \cdot x(t_K^-)$

$\Leftrightarrow \; x(t_K) = x(t_K^-) + B_K \cdot x(t_K^-) = (I+B_K)\,x(t_K^-)$

$$= x(t_K^-)^T \cdot [(I+B_K)^T P_{i_{K+1}} (I+B_K)] \cdot x(t_K^-)$$

$$\leq \lambda_{max}([I+B_K]^T P_{i_{K+1}} [I+B_K]) \cdot \| x(t_K^-) \|^2$$

$$\leq \lambda_{max}((I+B_K)^T (I+B_K)) \, \lambda_{max}(P_{i_{K+1}}) \| x(t_K^-) \|^2$$

$\lambda_{max}(P_{i_{K+1}}) \leq \max_{j \in J} (\lambda_{max}(P_{1j})) =: \lambda_{1M} \leq \mu \cdot \lambda_{1m}$

$$\leq \mu \cdot \lambda_{max}^2 (I+B_K) \; \lambda_{1m} \| x(t_K^-) \|^2$$

$\alpha_K = \mu \cdot \lambda_{max}^2(I+B_K)$, $\quad \lambda_{1m} = \min_{1 \leq j \leq N} (\lambda_{min}(P_{1j}))$

$\mu := \max \left\{ \frac{\lambda_{1M}}{\lambda_{1m}}, \frac{\lambda_{2M}}{\lambda_{2m}} \right\} \geq \frac{\lambda_{1M}}{\lambda_{1m}}$.

$$\leq \alpha_K \, \lambda_{min}(P_{1 i_K}) \cdot \| x(t_K^-) \|^2 \leq \alpha_K \, x(t_K^-)^T P_{1 i_K} x(t_K^-)$$

$x(t_K^-)^T P_{1 i_K} x(t_K^-) = V_{i_K}(t_K^-)$

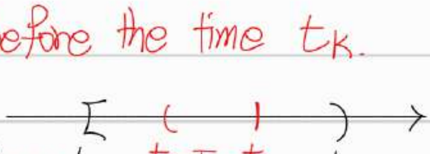
this term is completely determined by B_K. $\Rightarrow$ $V_{i_K}(t_K^-) \overset{B_K}{\longleftrightarrow} V_{i_{K+1}}(t_K)$

Similarly, we can show

$$W_{i_{K+1}}(t_K) \leq r_K W_{i_K}(t_K^-) + \beta_K V_{i_K}(t_K^-) + \psi_K \| (V_{i_K})_{t_K^-} \|_\tau$$

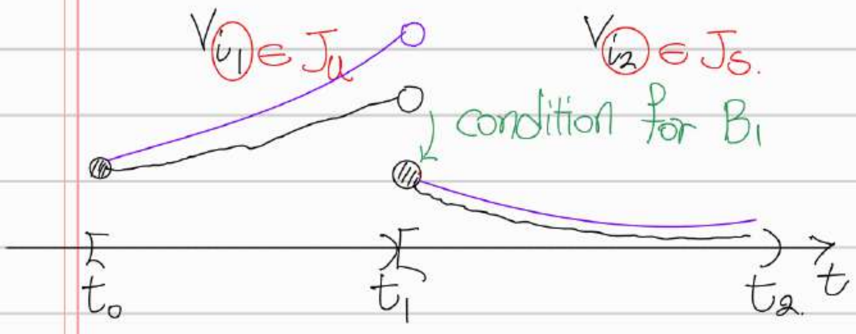
$\| (V_{i_K})_{t_K^-} \|_\tau = \lim_{t \to t_K^-} \| (V_{i_K})_t \|_\tau = \sup_{t-\tau \leq s \leq t < t_K} \| V_{i_K}(s) \|$

we want to control this using the information before the time t_K.



defs of r_K, β_K, ψ_K and Cauchy-Schwarts ineq. are used.

Let's consider the case where we run an unstable subsystem on the first interval and a stable one on the second interval.



$\sigma : [t_0, \infty) \to J = \{1, \dots, N\}$

$\sigma([t_{k-1}, t_k)) = i_k$

$\|(V_{i_2})_{t_1}\|_\tau := \sup_{t_0 \le t_1 - \tau \le t \le t_1} \|V_{i_2}(t)\|$

$t_1 - t_0 \ge \inf(t_k - t_{k-1}) \ge \tau > 0$

From def of $\mu := \max\left\{ \frac{\lambda_{1M}}{\lambda_{1m}}, \frac{\lambda_{2M}}{\lambda_{2m}} \right\}$

$\forall i, j \in J = [N]$, $V_i(t) \le \mu \cdot V_j(t)$ for $\forall t \in [t_0, \infty)$

① If $t \in [t_0, t_1) \Rightarrow \|V_{i_2}(t)\| = V_{i_2}(t) \le \mu \cdot V_{i_1}(t)$ (by def $V \ge 0$; not matched)

$\le \mu \cdot (\|(V_{i_1})_{t_0}\|_\tau + \|(W_{i_1})_{t_0}\|_\tau) \cdot e^{\xi_1(t - t_0)}$ by (*) ($i_1 \in J_u$)

$\le \underbrace{\mu}_{\ge 1} \cdot (\|(V_{i_1})_{t_0}\|_\tau + \|(W_{i_1})_{t_0}\|_\tau) \cdot e^{\xi_1(t_1 - t_0)}$ ∵ $t_0 \le t < t_1$

② If $t = t_1$, $\|V_{i_2}(t_1)\| = V_{i_2}(t_1) \le \alpha_k V_{i_1}(t_k^-)$ (matched well)

$\le \underbrace{\alpha_k}_{\ge 1 \, ?} (\|(V_{i_1})_{t_0}\|_\tau + \|(W_{i_1})_{t_0}\|_\tau) e^{\xi_1(t_1 - t_0)}$

$\Rightarrow \|(V_{i_2})_{t_1}\|_\tau := \sup_{t_1 - \tau \le t \le t_1} \|V_{i_2}(t)\| \le \alpha_1 \mu (\|(V_{i_1})_{t_0}\|_\tau + \|(W_{i_1})_{t_0}\|_\tau) e^{\xi_1(t_1 - t_0)}$

Similarly, we can show $\|(W_{i_2})_{t_1}\| \le \mu(\beta_1 + \gamma_1 + \psi_1)(\|(V_{i_1})_{t_0}\|_\tau + \|(W_{i_1})_{t_0}\|_\tau) e^{\xi_1(t_1 - t_0)}$

From this point, I have made some modifications

Generally, we have

$$V_{i_n}(t) \le \underbrace{\prod_{\substack{1\le k<n \\ i_k\in J_u}} \mu\cdot(\alpha_k+\beta_k+\gamma_k+\psi_k)}_{\le \prod_{i_k\in J_u} e^{\nu(t_k-t_{k-1})}} e^{\overset{\le \lambda^+}{\xi_k}(t_k-t_{k-1})}$$

$\forall t>t_0$, $\exists!\, n\in\mathbb{N}$ s.t. $t\in[t_{n-1}, t_n)$

$$\times \underbrace{\prod_{\substack{1\le l<n \\ i_l\in J_s}} \mu\cdot(\alpha_l+\beta_l+\gamma_l+\psi_l\, e^{\xi_l\tau})e^{\xi_l\tau}}_{\le \prod_{i_l\in J_s} e^{\nu(t_k-t_{k-1})}} e^{-\overset{\le -\lambda^-}{\xi_l}(t_l-t_{l-1})} \quad ??$$

$$\times (\|(V_{i_1})_{t_0}\|_\tau + \|(W_{i_1})_{t_0}\|_\tau)\, e^{\begin{cases}\xi_n(t-t_{n-1}) & \text{if } i_n\in J_u \\ -\xi_n(t-t_{n-1}) & \text{if } i_n\in J_s\end{cases}}$$

$$\le e^{\begin{cases}\lambda^+(t-t_{n-1}) & (\text{''}) \\ -\lambda^-(t-t_{n-1}) & (\text{''})\end{cases}}$$

$$\overset{(A4)}{\le} e^{\nu(t_{n-1}-t_0)} \times e^{\lambda^+\cdot T^+(t_0,t)} \times e^{-\lambda^- T^-(t_0,t)} \times \text{▨}$$

$$\le e^{\nu(t-t_0)} \times e^{-\lambda^*(t-t_0)} \times \text{▨}$$

($\ge$ since $t_{n-1}\le t$; $\ge$ by ㉠)

㉠ : $\forall t>t_0$, $\dfrac{T^-(t_0,t)}{T^+(t_0,t)} \ge \inf_{s\ge t_0} \dfrac{T^-(t_0,s)}{T^+(t_0,s)} \overset{(A4)}{\ge} \dfrac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*}$

$\Rightarrow$ Since $\lambda^- - \lambda^* > 0$ ($\because \lambda^*\in(0,\lambda^-)$),

$\lambda^- T^- - \lambda^* T^- \ge T^+\lambda^+ + T^+\lambda^*$

$\Leftrightarrow \lambda^+ T^+(t_0,t) - \lambda^- T^-(t_0,t) \le -\lambda^*(\underbrace{T^-(t_0,t)+T^+(t_0,t)}_{= t-t_0 \text{ by def.}})$

$$\le e^{-(\lambda^*-\nu)(t-t_0)} \times (\|(V_{i_1})_{t_0}\|_\tau + \|(W_{i_1})_{t_0}\|_\tau)$$

$$\Rightarrow \|x(t)\|^2 \le \frac{1}{\lambda_{lm}} V_{i_n}(t) \le \frac{1}{\lambda_{lm}}\cdot \text{▨}$$

$$\Rightarrow \exists K_1>0 \text{ s.t. } \|x(t)\| \le K_1(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau)\, e^{-(\lambda^*-\nu)(t-t_0)/2}$$

Similarly, we have

$$\sup_{t_0-\tau\le s\le t_0} V_i(S) = x(s)^T P_{i_1} x(s).$$

$$W_{i_n}(t) \le \left(\|(V_{i_1})_{t_0}\|_\tau + \|(W_{i_1})_{t_0}\|_\tau \right) e^{-(\lambda^* - \nu)(t-t_0)}$$

nth time interval $[t_{n-1}, t_n) \ni t$

By using the fact that

$$\|z\| - \|h_{i_n}\| \le \|z - h_{i_n}\| \le \frac{1}{\sqrt{\lambda_{2m}}} W_{i_n}^{1/2}$$

We can see $\exists K_2$ s.t.

$$\|z(t)\| \le K_2 \left(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau \right) e^{-(\lambda^* - \nu)(t-t_0)/2}$$

$$\therefore \|x(t)\| + \|z(t)\| \le K \left(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau \right) e^{-(\lambda^* - \nu)(t-t_0)/2}$$